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Euler equations on a semi-direct product of the diffeomorphisms group by itself

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Abstract

The geodesic equations of a class of *right invariant* metrics on the semi-direct product $\mathrm{Diff}(\mathbb{S}^1)\circledS\mathrm{Diff}(\mathbb{S}^1)$ are studied. The equations are explicitly described, they have the form of a system of coupled equations of Camassa-Holm type and possess singular (peakon) solutions. Their integrability is further investigated, however no compatible bi-Hamiltonian structures on the corresponding dual Lie algebra $(\mathrm{Vect}(\mathbb{S}^1)\circledS\mathrm{Vect}(\mathbb{S}^1))^*$ are found.

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1 Introduction

V.I. Arnold put forward the idea of using geodesic flow in the analysis of the motion of hydrodynamical systems [1]. He showed that the Euler equations of hydrodynamics (with fixed boundary) can be obtained as geodesic equations of a right invariant Riemannian metric on the group of volume-preserving diffeomorphisms. This structure is a prototype for the mathematical treatment of many physical systems when the configuration space can be identified with a Lie group. Such examples are: the Euler pegtop as a geodesic equation for the metric[1] $\langle \omega, I\omega \rangle$ on the $SO(3)$ group;

[1] Here $I : SO(3) \to SO(3)^*$ denotes the inertia matrix of the body.

the KdV equation as a geodesic equation for the L^2 metric on the Virasoro group; Camassa-Holm equation [5] as a geodesic equation for the H^1 metric on the Virasoro group (or $\mathrm{Diff}(\mathbb{S}^1)$)) etc.

Both KdV and Camassa-Holm (CH) equations are integrable models for propagation of shallow water waves [5, 24, 13, 22, 19]. The CH equation was extensively studied in the recent years and many of the underlying qualitative aspects (e.g. the identification of initial profiles that develop into global solutions or into breaking waves [3, 4, 6, 7, 8, 18] rely on properties arising from its geometric structure as a geodesic flow equation [27, 21, 10, 11, 12].

There are also examples of hydrodynamic equations which are geodesic flows for metrics on semidirect products of Lie groups [2, 21, 9, 23, 29], such as the barotropic fluid equation (see e.g. [2]) and the two-component integrable shallow water equation of CH type (see e.g. [15, 9, 23, 29]).

In this paper we study the geodesic flow equations of a class of right invariant metrics on $\mathrm{Diff}(\mathbb{S}^1)\circledS\mathrm{Diff}(\mathbb{S}^1)$. We begin with the general framework for the Euler equation on a Lie group and introduce the notion of a bi-Hamiltonian Euler equation in Section 2. Then we extend this construction to a semidirect product of Lie groups in Section 3. In particular we investigate families of Euler equations on the semidirect product of $\mathrm{Diff}(\mathbb{S}^1)$ by itself (twisted by the inner representation) in Section 4. The integrability of the obtained equations is further investigated in Section 5.

2 Euler equation on a Lie group

On a general Riemannian manifold M, the geodesic flow is a *Hamiltonian flow* for the *canonical symplectic structure* on the cotangent bundle T^*M. The Hamiltonian is given by the energy functional. When M is a Lie group G, the canonical symplectic structure is invariant (either by left or right translations). It induces a *Poisson structure* on the quotient space

$$T^*G/G \simeq \mathfrak{g}^*$$

called the *Lie-Poisson structure*

$$\{H, K\}(m) = m([d_mH, d_mK]), \qquad H, K \in C^\infty(\mathfrak{g}^*).$$

Notice that d_mH, d_mK are elements of the Lie algebra $\mathfrak{g}$ so that the preceding formula is meaningful.

A right invariant metric on G is completely determined by its value at the unit element e of the group, that is, by an inner product on its Lie algebra $\mathfrak{g}$. This inner product can be expressed in terms of a symmetric linear operator, called the *inertia operator*

$$A : \mathfrak{g} \to \mathfrak{g}^*.$$

The (invariant) energy functional generates a reduced Hamiltonian function on $\mathfrak{g}^*$

$$H(m) = \frac{1}{2}\,(m, A^{-1}m).$$

Its differential at m is $d_mH = A^{-1}m \in \mathfrak{g}$ and its corresponding Hamiltonian vector field on $\mathfrak{g}^*$ is

$$X(m) = P_m(A^{-1}m), \qquad m \in \mathfrak{g}^*$$

where P is the Lie-Poisson bivector. The corresponding evolution equation is known as the *Euler equation* of the invariant metric.

Sometimes, this Euler equation is in fact *bi-Hamiltonian* relatively to some modified Lie-Poisson structure. Under the general name of *modified Lie-Poisson structures*, we mean an affine[2] perturbation of the canonical Lie-Poisson structure on $\mathfrak{g}^*$. In other words, it is represented by a *Poisson bivector*

$$P + Q,$$

where P is the canonical Poisson bivector and Q is a constant bivector on $\mathfrak{g}^*$. Such a $Q \in \bigwedge^2 \mathfrak{g}^*$ is indeed a Poisson bivector, since the Schouten-Nijenhuis bracket [28]

$$[Q, Q] = 0,$$

for a constant tensor field on $\mathfrak{g}^*$.

The fact that $P + Q$ is a Poisson bivector, or equivalently that Q is compatible with the Lie-Poisson structure

$$[P, Q] = 0,$$

is expressed by the condition

$$Q([u, v], w) + Q([v, w], u) + Q([w, u], v) = 0, \tag{1}$$

for all $u, v, w \in \mathfrak{g}$. In other words, Q is a *2-cocycle* of the Lie algebra $\mathfrak{g}$.

A special case is when Q is a coboundary. In that case

$$Q(u, v) = (\partial m_0)(u, v) = m_0([u, v])$$

for some $m_0 \in \mathfrak{g}^*$ and the expression

$$\{f, g\}_0(m) = m_0([d_m f, d_m g])$$

looks like if the Lie-Poisson bracket had been "frozen" at a point $m_0 \in \mathfrak{g}^*$. For this reason it is sometimes called a *freezing* structure [2, 25].

3 Semi-direct products of Lie groups

We consider now an abstract Lie group G and its left action on itself by conjugacy:

$$g \cdot h = ghg^{-1}.$$

This defines a semi-direct product on $G \times G$:

$$(g_1, h_1) \star (g_2, h_2) = \big(g_1 g_2, h_1 (g_1 \cdot h_2)\big) = \big(g_1 g_2, h_1 g_1 h_2 g_1^{-1}\big).$$

The inverse of an element (g, h) is given by:

$$(g, h)^{-1} = (g^{-1}, g^{-1} h^{-1} g).$$

[2] A Poisson structure on a linear space is *affine* if the bracket of two linear functionals is an affine functional.

The inner action of the semi-direct product $G \circledS G$ on itself is given by:

$$\begin{aligned} I_{(g_1,h_1)}(g_2,h_2) &= (g_1,h_1) \star (g_2,h_2) \star (g_1,h_1)^{-1} \\ &= \big(g_1 g_2 g_1^{-1}, h_1 g_1 (h_2 g_2)(h_1 g_1)^{-1} g_1 g_2^{-1} g_1^{-1}\big) \end{aligned}$$

From it, we deduce the adjoint action of $G \circledS G$ on its Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$:

$$\mathrm{Ad}_{(g_1,h_1)}(u_2,v_2) = \big(\, \mathrm{Ad}_{g_1} u_2, \mathrm{Ad}_{h_1 g_1} u_2 + \mathrm{Ad}_{h_1 g_1} v_2 - \mathrm{Ad}_{g_1} u_2\big)$$

and the adjoint action of $\mathfrak{g} \oplus \mathfrak{g}$ on itself:

$$\mathrm{ad}_{(u_1,v_1)}(u_2,v_2) = \big(\, \mathrm{ad}_{u_1} u_2, \mathrm{ad}_{v_1} u_2 + \mathrm{ad}_{u_1} v_2 + \mathrm{ad}_{v_1} v_2\big).$$

in particular, we obtain the Lie bracket on the Lie algebra $\mathfrak{g} \circledS \mathfrak{g}$ of the semi-direct product $G \circledS G$:

$$[(u_1,v_1),(u_2,v_2)] = \big([u_1,u_2],[v_1,u_2] + [u_1,v_2] + [v_1,v_2]\big) \tag{2}$$

4 Euler equations on $\mathrm{Diff}(\mathbb{S}^1) \circledS \mathrm{Diff}(\mathbb{S}^1)$

In this section, we apply the theory for $G = \mathrm{Diff}(\mathbb{S}^1)$ the diffeomorphisms group of the circle. Its *Lie algebra* $\mathrm{Vect}(\mathbb{S}^1)$ is isomorphic to $\mathrm{C}^\infty(\mathbb{S}^1)$ with the Lie bracket given by

$$[u,v] = u_x v - u v_x.$$

Note that it corresponds to the opposite of the usual Lie bracket of vector fields.

The *regular dual* $\mathrm{Vect}^*(\mathbb{S}^1)$ of $\mathrm{Vect}(\mathbb{S}^1)$ is defined as the subspace of linear functionals of the form

$$u \mapsto \int_{\mathbb{S}^1} m u \, dx$$

for some function $m \in \mathrm{C}^\infty(\mathbb{S}^1)$.

Let F be a smooth real valued function on $\mathrm{C}^\infty(\mathbb{S}^1)$. Its *Fréchet* derivative $dF(m)$ is a linear functional on $\mathrm{C}^\infty(\mathbb{S}^1)$. We say that F is a *regular function* if there exists a smooth map $\delta F : \mathrm{C}^\infty(\mathbb{S}^1) \to \mathrm{C}^\infty(\mathbb{S}^1)$ such that

$$dF(m)v = \int_{\mathbb{S}^1} v \left(\frac{\delta F}{\delta m}\right) dx, \quad m,\, v \in \mathrm{C}^\infty(\mathbb{S}^1).$$

The map $\delta F/\delta m$ is the L^2*-gradient* of F. Note that the second derivative of a regular function is symmetric [17] and therefore the derivative of a gradient is a L^2-symmetric operator.

The canonical Lie-Poisson structure on $\mathrm{Vect}^*(\mathbb{S}^1)$ is given by

$$\{F, G\}(m) = \int_{\mathbb{S}^1} \left(\frac{\delta F}{\delta m}\right) P_m \left(\frac{\delta G}{\delta m}\right) dx$$

where $P_m = -(mD + Dm)$ and $D := d/dx$.

The theory extends straightforwardly to the case of $(\mathrm{Vect}(\mathbb{S}^1) \circledS \mathrm{Vect}(\mathbb{S}^1))^*$. The differential of a regular function F is given by

$$dF(m,n)(v,w) = \int_{\mathbb{S}^1} \left[v \left(\frac{\delta F}{\delta m}\right) + w \left(\frac{\delta F}{\delta n}\right) \right] dx.$$

The canonical Lie-Poisson structure on $(\mathrm{Vect}(\mathbb{S}^1) ⓢ \mathrm{Vect}(\mathbb{S}^1))^*$ is given by

$$\{F, G\}(m, n) = \int_{\mathbb{S}^1} \left(\frac{\delta F}{\delta m}, \frac{\delta F}{\delta n}\right) \mathbb{P}_{(m,n)} \left(\frac{\delta G}{\delta m}, \frac{\delta G}{\delta n}\right) dx$$

where

$$\mathbb{P}_{(m,n)} = \begin{pmatrix} P_m & P_n \\ P_n & P_n \end{pmatrix}$$

where $P_m = -(mD + Dm)$.

A *right invariant* metric on $\mathrm{Diff}(\mathbb{S}^1) ⓢ \mathrm{Diff}(\mathbb{S}^1)$ is defined by an inner product $\mathbf{a}$ on the Lie algebra of the group $\mathrm{C}^\infty(\mathbb{S}^1) \oplus \mathrm{C}^\infty(\mathbb{S}^1)$. If this inner product is *local*, it is given by

$$\mathbf{a}\big((u_1, v_1), (u_2, v_2)\big) = \int_{\mathbb{S}^1} (u_1, v_1)\mathbb{A}(u_2, v_2) dx \qquad u, v \in \mathrm{C}^\infty(\mathbb{S}^1),$$

where

$$\mathbb{A} = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$$

is a symmetric linear differential operator on $\mathrm{C}^\infty(\mathbb{S}^1) \oplus \mathrm{C}^\infty(\mathbb{S}^1)$. The corresponding *Euler vector field* is given by

$$X(m, n) = \mathbb{P}_{(m,n)}\mathbb{A}^{-1}(m, n).$$

In the sequel, we will suppose that

$$\mathbb{A} = (1 - D^2) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \tag{3}$$

for some real constants a, b and c such that $ab - c^2 \neq 0$.

The corresponding Euler equation is then

$$\begin{cases} m_t = -2mu_x - m_x u - 2nv_x - n_x v \\ n_t = -2n(u_x + v_x) - n_x(u + v) \end{cases} \tag{4}$$

where

$$\begin{pmatrix} m \\ n \end{pmatrix} = \mathbb{A} \begin{pmatrix} u \\ v \end{pmatrix}$$

Introducing new variables $M = m - n$, $V = u + v$, $U = u$ and $N = n$ we write (4) as

$$\begin{cases} M_t = -2MU_x - M_x U \\ N_t = -2NV_x - N_x V \end{cases} \tag{5}$$

In general, we have a linear relation between the variables, i.e.

$$\begin{pmatrix} M \\ N \end{pmatrix} = (1 - D^2) \begin{pmatrix} M_0 \cos\alpha & M_0 \sin\alpha \\ N_0 \sin\beta & N_0 \cos\beta \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \tag{6}$$

for some real parameters M_0, N_0, α and β. The consistency with (6) gives

$$b = N_0 \cos\beta, \qquad c - b = N_0 \sin\beta = M_0 \sin\alpha,$$

or

$$\begin{pmatrix} M \\ N \end{pmatrix} = (N_0 \sin\beta)(1-D^2) \begin{pmatrix} \cot\alpha & 1 \\ 1 & \cot\beta \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \tag{7}$$

With an overall rescaling of M and N by the constant $N_0 \sin\beta$ we can remove this overall factor from (7) and thus, there are only two parameters, α and β relating (M, N) and (U, V). Since the equations (5) allow rescaling of both M and N by a constant, for convenience we are going to write from now on

$$\begin{cases} M = \cos\alpha(U - U_{xx}) + \sin\alpha(V - V_{xx}), \\ N = \sin\beta(U - U_{xx}) + \cos\beta(V - V_{xx}). \end{cases} \tag{8}$$

There are two special subcases of the system (5), (8): If $\alpha = \beta = 0$ it reduces to two independent Camassa-Holm equations; if $\alpha = \beta = \pi/2$ the case is a cross-coupled Camassa-Holm system, studied in [14]. The cross-coupled system admits peakon solutions with an interesting behavior, investigated in the above article, which can be visually described as "waltzing peakons".

In fact, the system (5), (8) admits peakon solutions for any values of the parameters α and β such that $\alpha + \beta \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \ldots$. Indeed, the peakon Ansatz (which is also a singular momentum map, [20])

$$\begin{cases} M = \displaystyle\sum_{k=1}^{K} M_k \delta(x - q_k), \\ N = \displaystyle\sum_{l=1}^{L} N_l \delta(x - r_l) \end{cases} \tag{9}$$

together with (8)

$$\begin{cases} U = \dfrac{1}{2\cos(\alpha+\beta)} \left(\cos\beta \displaystyle\sum_{k=1}^{K} M_k e^{-|x-q_k|} - \sin\alpha \sum_{l=1}^{L} N_l e^{-|x-r_l|} \right), \\ V = \dfrac{1}{2\cos(\alpha+\beta)} \left(-\sin\beta \displaystyle\sum_{k=1}^{K} M_k e^{-|x-q_k|} + \cos\alpha \sum_{l=1}^{L} N_l e^{-|x-r_l|} \right) \end{cases} \tag{10}$$

gives the following dynamical system for the peakon parameters:

$$
\begin{aligned}
\dot{M}_k &= \frac{M_k}{2\cos(\alpha+\beta)}\times \\
&\left(\cos\beta\sum_{p=1}^{K} M_p e^{-|q_k-q_p|}\mathrm{sgn}(q_k-q_p) - \sin\alpha\sum_{p=1}^{L} N_p e^{-|q_k-r_p|}\mathrm{sgn}(q_k-r_p)\right), \\
\dot{q}_k &= \frac{1}{2\cos(\alpha+\beta)}\left(\cos\beta\sum_{p=1}^{K} M_p e^{-|q_k-q_p|} - \sin\alpha\sum_{p=1}^{L} N_p e^{-|q_k-r_p|}\right), \\
\dot{N}_l &= \frac{N_l}{2\cos(\alpha+\beta)}\times \\
&\left(-\sin\beta\sum_{p=1}^{K} M_p e^{-|r_l-q_p|}\mathrm{sgn}(r_l-q_p) + \cos\alpha\sum_{p=1}^{L} N_p e^{-|r_l-r_p|}\mathrm{sgn}(r_l-r_p)\right), \\
\dot{r}_l &= \frac{1}{2\cos(\alpha+\beta)}\left(-\sin\beta\sum_{p=1}^{K} M_p e^{-|r_l-q_p|} + \cos\alpha\sum_{p=1}^{L} N_p e^{-|r_l-r_p|}\right).
\end{aligned} \tag{11}
$$

Note that the peakon parameters depend only on t and the singular solutions of the geodesic equations are uniquely determined by these parameters via (9).

5 Integrability

The interesting question is now : is this equation bi-Hamiltonian for some modified Lie-Poisson structure?

Remark first that a freezing structure on $(\mathrm{Vect}(\mathbb{S}^1)Ⓢ\mathrm{Vect}(\mathbb{S}^1))^*$ corresponds to the following operator

$$
\begin{pmatrix} P_{m_0} & P_{n_0} \\ P_{n_0} & P_{n_0} \end{pmatrix}
$$

where m_0, n_0 are fixed elements of $\mathrm{C}^\infty(\mathbb{S}^1)$. If m_0 and n_0 are constant functions, this operator writes down as

$$
\begin{pmatrix} \alpha D & \beta D \\ \beta D & \beta D \end{pmatrix},
$$

where $\alpha, \beta \in \mathbb{R}$. Notice furthermore that even *without computing the whole Gel'fand-Fuks cohomology* [16] of $\mathrm{Vect}(\mathbb{S}^1)Ⓢ\mathrm{Vect}(\mathbb{S}^1)$, we can check that the following skew-adjoint operator

$$
\begin{pmatrix} \gamma D^3 & \delta D^3 \\ \delta D^3 & \delta D^3 \end{pmatrix}
$$

where $\gamma, \delta \in \mathbb{R}$, is a 2-cocycle; i.e it satisfies the cocycle condition (1). Moreover, it is not a *coboundary* [26].

Now, recall the following criterion which is a natural extension to $\mathrm{C}^\infty(\mathbb{S}^1) \oplus \mathrm{C}^\infty(\mathbb{S}^1)$ of a result established in [12] for $\mathrm{C}^\infty(\mathbb{S}^1)$.

Proposition 1 *A necessary condition for a smooth vector field X on $\mathrm{C}^\infty(\mathbb{S}^1) \oplus \mathrm{C}^\infty(\mathbb{S}^1)$ to be* Hamiltonian *with respect to the Poisson structure defined by a* constant *linear operator Q is the symmetry of the operator $X'(m,n)Q$ for each $(m,n) \in \mathrm{C}^\infty(\mathbb{S}^1) \oplus \mathrm{C}^\infty(\mathbb{S}^1)$.*

The vector field $X(m,n)$ defined in (4) can be written as

$$X(m,n) = \begin{pmatrix} P_m(u) + P_n(v) \\ P_n(u) + P_n(v) \end{pmatrix}$$

where $P_m(u) = -(2mu_x + m_x u)$,

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \Lambda(u) \\ \Lambda(v) \end{pmatrix},$$

and $\Lambda = 1 - D^2$. Therefore, the components of its the Fréchet derivative are given by

$$\begin{aligned}
X'(m,n)[1,1] &= \frac{1}{d}\left(P_{cm-bn}\Lambda^{-1} - d(P_u + 3Du)\right) \\
X'(m,n)[1,2] &= \frac{1}{d}\left(P_{an-bm}\Lambda^{-1} - d(P_v + 3Dv)\right) \\
X'(m,n)[2,1] &= \frac{1}{d}\left(P_{(b-c)n}\Lambda^{-1}\right) \\
X'(m,n)[2,2] &= \frac{1}{d}\left(P_{(b-a)n}\Lambda^{-1} - d(P_u + P_v + 3Du + 3Dv)\right)
\end{aligned}$$

where $d = ac - b^2$. We are going to check if there exists m_0, n_0, γ, δ such that $X'(m,n)Q$ is symmetric, where

$$Q = \begin{pmatrix} P_{m_0} + \gamma D^3 & P_{n_0} + \delta D^3 \\ P_{n_0} + \delta D^3 & P_{n_0} + \delta D^3 \end{pmatrix},$$

that is

$$p^* = p, \qquad q^* = r, \qquad s^* = s \tag{12}$$

where

$$\begin{aligned}
p &= \left(P_{cm-bn}\Lambda^{-1} - d(P_u + 3Du)\right)(P_{m_0} + \gamma D^3) \\
&\quad + \left(P_{an-bm}\Lambda^{-1} - d(P_v + 3Dv)\right)(P_{n_0} + \delta D^3), \\
q &= \left(P_{cm-bn}\Lambda^{-1} - d(P_u + 3Du)\right)(P_{n_0} + \delta D^3) \\
&\quad + \left(P_{an-bm}\Lambda^{-1} - d(P_v + 3Dv)\right)(P_{n_0} + \delta D^3), \\
r &= \left(P_{(b-c)n}\Lambda^{-1}\right)(P_{m_0} + \gamma D^3) \\
&\quad + \left(P_{(b-a)n}\Lambda^{-1} - d(P_u + P_v + 3Du + 3Dv)\right)(P_{n_0} + \delta D^3), \\
s &= \left(P_{(b-c)n}\Lambda^{-1}\right)(P_{n_0} + \delta D^3) \\
&\quad + \left(P_{(b-a)n}\Lambda^{-1} - d(P_u + P_v + 3Du + 3Dv)\right)(P_{n_0} + \delta D^3).
\end{aligned}$$

Taking first $u = 1$ and $v = 0$ in $p^* = p$ and applying to the constant function 1, we get that $m_0 =$ constant and therefore $P_{m_0} = \lambda D$ for some $\lambda \in \mathbb{R}$.

Similarly, taking $u = 0$ and $v = 1$ in $p^* = p$ and applying to the constant function 1, we get $P_{n_0} = \mu D$ for some $\mu \in \mathbb{R}$.

The preceding equations recast thus as

$$\begin{aligned}
p &= \left(P_{cm-bn}\Lambda^{-1} - d(P_u + 3Du)\right)(\lambda D + \gamma D^3) \\
&\quad + \left(P_{an-bm}\Lambda^{-1} - d(P_v + 3Dv)\right)(\mu D + \delta D^3), \\
q &= \left(P_{cm-bn}\Lambda^{-1} - d(P_u + 3Du)\right)(\mu D + \delta D^3) \\
&\quad + \left(P_{an-bm}\Lambda^{-1} - d(P_v + 3Dv)\right)(\mu D + \delta D^3), \\
r &= \left(P_{(b-c)n}\Lambda^{-1}\right)(\lambda D + \gamma D^3) \\
&\quad + \left(P_{(b-a)n}\Lambda^{-1} - d(P_u + P_v + 3Du + 3Dv)\right)(\mu D + \delta D^3), \\
s &= \left(P_{(b-c)n}\Lambda^{-1}\right)(\mu D + \delta D^3) \\
&\quad + \left(P_{(b-a)n}\Lambda^{-1} - d(P_u + P_v + 3Du + 3Dv)\right)(\mu D + \delta D^3).
\end{aligned}$$

Taking then $n = 0$ in $s^* = s$ and applying it to the constant function 1 leads to $\mu = \delta = 0$ or $b = c$. But then we get that $r^*(1) = 0$ which leads to $\lambda = \mu = 0$ if $b \neq c$.

If $b = c$, there are solutions to equations (12), namely

$$Q = \gamma \begin{pmatrix} D\Lambda & 0 \\ 0 & 0 \end{pmatrix}$$

where $\gamma \in \mathbb{R}$. However, for the vector field X to be Hamiltonian with respect to the Poisson structure defined by the Poisson bivector Q, that is

$$X(m,n) = QdH(m,n)$$

for some Hamiltonian function H, it is necessary that X belongs to the range of Q. This would require that $P_n(u+v) = 0$, for all $m, n \in \mathrm{C}^\infty(\mathbb{S}^1)$ which is not the case for values a, b such that $d = ab - b^2 \neq 0$.

This shows that for there is no non-trivial modified structure given by

$$Q = \begin{pmatrix} P_{m_0} + \gamma D^3 & P_{n_0} + \delta D^3 \\ P_{n_0} + \delta D^3 & P_{n_0} + \delta D^3 \end{pmatrix},$$

which makes (4) bi-Hamiltonian.

6 Conclusion

In this paper, we have presented the geometric approach of constructing Euler equations associated to the metrics on $\mathrm{Diff}(\mathbb{S}^1)$Ⓢ$\mathrm{Diff}(\mathbb{S}^1)$. From the point of view of mathematical physics this construction leads to a family of coupled peakon equations, some of which are known to have interesting behaviour e.g. the system of coupled Camassa-Holm type equations without self-interactions where each of the two types of peakon solutions moves only under the induced velocity of the other type. As a result a 'waltzing' solution behaviour is observed, see for example [14] for some interesting figures and movies. This example is an indication that the other equations of the family also have interesting properties that remain to be studied.

An open question remains the integrability of the obtained equations. Our search for compatible bi-Hamiltonian structures on the dual Lie algebra $(\mathrm{Vect}(\mathbb{S}^1)$Ⓢ$\mathrm{Vect}(\mathbb{S}^1))^*$ in Section 5 does not give a positive result, which however does not exclude other possibilities for compatible bi-Hamiltonian structures.

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References

[1] V. I. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier (Grenoble), **16** (fasc. 1) (1966), 319–361.

[2] V. I. Arnold and B. Khesin, "Topological methods in hydrodynamics", Applied Mathematical Sciences, vol. 125, Springer-Verlag, New York, 1998.

[3] A. Bressan and A. Constantin, *Global dissipative solutions of the Camassa-Holm equation*, Anal. Appl. (Singapore) **5** (2007), 1–27.

[4] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equation*, Arch. Ration. Mech. Anal. **183** (2007), 215–239.

[5] R. Camassa and D.D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett., **71** (1993), 1661–1664.

[6] A. Constantin, *Existence of permanent and breaking waves for a shallow water equation: a geometric approach*, Ann. Inst. Fourier (Grenoble) **50** (2000), 321–362.

[7] A. Constantin and J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math., **181** (1998), 229–243.

[8] A. Constantin and J. Escher, *On the blow-up rate and the blow-up set of breaking waves for a shallow water equation*, Math. Z. **233** (2000), 75–91.

[9] A. Constantin and R. Ivanov, *On an integrable two-component Camassa-Holm shallow water system*, Phys. Lett. A **372** (2008), 7129–7132.

[10] A. Constantin and B. Kolev, *On the geometric approach to the motion of inertial mechanical systems*, J. Phys. A, **35** (2002), R51–R79.

[11] A. Constantin and B. Kolev, *Geodesic flow on the diffeomorphism group of the circle*, Comment. Math. Helv., **78** (2003), 787–804.

[12] A. Constantin and B. Kolev, *Integrability of invariant metrics on the diffeomorphism group of the circle*, J. Nonlinear Sci. **16** (2006), 109–122.

[13] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi Equations*, Arch. Rat. Mech. Anal., **192** (2009), 165–186.

[14] C. Cotter, D. Holm, R I. Ivanov and J. R. Percival, *Waltzing peakons and compacton pairs in a cross-coupled Camassa-Holm equation*, J. Phys. A: Math. Theor., **44** (2011), 265–205.

[15] J. Escher, *Non-metric two-component Euler equations on the circle*, Monatshefte für Mathematik, published online: 23 June 2011, DOI: 10.1007/s00605-011-0323-3.

[16] (MR0245035) I. M. Gel'fand and, D. B. Fuks,*Cohomologies of the Lie algebra of vector fields on the circle*, Funkcional. Anal. i Prilovzen. **2** (1968), no. 4, 92–93.

[17] R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 65–222.

[18] D. J. Henry, *Compactly supported solutions of the Camassa-Holm equation*, J. Nonlinear Math. Phys., **12** (2005), 342–347.

[19] D. D. Holm and R. I. Ivanov, *Smooth and peaked solitons of the Camassa-Holm equation and applications*, J. of Geometry and Symmetry in Physics, **22** (2011), 13–49.

[20] D. D. Holm and J. E.Marsden, *Momentum maps and measure-valued solutions (peakons, filaments and sheets) for the EPDiff equation*, in "The Breadth of Symplectic and Poisson Geometry" (eds. J. E. Marsden and T. S. Ratiu), Progr. Math. vol. 232, Birkhäuser Boston, Boston, MA, (2004), 203–235.

[21] D. D. Holm, J. E. Marsden and T. S. Ratiu, *The Euler-Poincaré Equations and Semidirect Products with Applications to Continuum Theories*, Adv. Math., **137** (1998), 1–81.

[22] R. I. Ivanov, *Water waves and integrability*, Philos. Trans. Roy. Soc. Lond. Ser. A. Math. Phys. Eng. Sci., **365** (2007), 2267–2280.

[23] R. I. Ivanov, *Two-component integrable systems modelling shallow water waves: the constant vorticity case*, Wave Motion **46** (2009), 389–396.

[24] R. S. Johnson, *Camassa-Holm, Korteweg-de Vries and Related Models for Water Waves*, J. Fluid Mech., **455** (2002), 63–82.

[25] B. Khesin and G. Misiołek, *Euler equations on homogeneous spaces and Virasoro orbits*, Adv. Math., **176** (2003), 116–144.

[26] B. Kolev, *Bi-Hamiltonian systems on the dual of the Lie algebra of vector fields of the circle and periodic shallow water equations*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **365** (2007), 2333–2357.

[27] G. Misiołek, *A shallow water equation as a geodesic flow on the Bott-Virasoro group*, J. Geom. Phys., **24** (1998), 203–208.

[28] I. Vaisman, "Lectures on the geometry of Poisson manifolds", Progress in Mathematics , vol. 118, Birkhäuser Verlag, Basel, 1994.

[29] P. Zhang and Y. Liu, *Stability of solitary waves and wave-breaking phenomena for the two-component Camassa-Holm system*, Int. Math. Res. Not. IMRN 2010, no. 11, 1981–2021.